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Generalized Quadrangles of Order (s, s^2), I

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In this paper generalized quadrangles of order (s, s^2), $s > 1$, satisfying property (G) at a line, at a pair of points, or at a flag, are studied. Property (G) was introduced by S. E. Payne (*Geom. Dedicata* **32** (1989), 93–118) and is weaker than 3-regularity (see S. E. Payne and J. A. Thas, "Finite Generalized Quadrangles," Pitman, London, 1984). It was shown by Payne that each generalized quadrangle of order (s^2, s), $s > 1$, arising from a flock of a quadratic cone, has property (G) at its point (∞). In particular translation generalized quadrangles satisfying property (G) are considered here. As an application it is proved that the Roman generalized quadrangles of Payne contain at least $s^3 + s^2$ classical subquadrangles $Q(4, s)$. Also, as a by-product, several classes of ovoids of $Q(4, s)$, s odd, are obtained; one of these classes is new. The goal of Part II is the classification of all translation generalized quadrangles satisfying property (G) at some flag $((\infty), L)$.

1. GENERALIZED QUADRANGLES

1.1. *Introduction*

A (finite) generalized quadrangle (GQ) is an incidence structure $\mathcal{S} = (P, B, \mathrm{I})$ in which P and B are disjoint (nonempty) sets of objects called points and lines (respectively), and for which I is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1 + t$ lines $(t \geqslant 1)$ and two distinct points are incident with at most one line.

(ii) Each line is incident with $1 + s$ points $(s \geqslant 1)$ and two distinct lines are incident with at most one point.

(iii) If x is a point and L is a line not incident with x, then there is a unique pair $(y, M) \in P \times B$ for which $x \mathrm{I} M \mathrm{I} y \mathrm{I} L$.

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The integers s and t are the parameters of the GQ and $\mathcal{S}$ is said to have order (s, t); if $s = t$, $\mathcal{S}$ is said to have order s. There is a point-line duality for GQ for which in any definition or theorem the words "point" and "line" are interchanged and the parameters s and t are interchanged. Normally, we assume without further notice that the dual of a given theorem or definition has also been given.

Given two (not necessarily distinct) points x, y of the GQ $\mathcal{S}$, we write $x \sim y$ and say that x and y are collinear, provided that there is some line L for which $x \mathrm{I} L \mathrm{I} y$. And $x \not\sim y$ means that x and y are not collinear. Dually, for $L, M \in B$, we write $L \sim M$ or $L \not\sim M$ according as L and M are concurrent or nonconcurrent, respectively. If $x \sim y$ (resp., $L \sim M$) we may also say that x (resp., L) is orthogonal or perpendicular to y (resp., M). The line (resp., point) which is incident with distinct collinear points x, y (resp. distinct concurrent lines L, M) is denoted by xy (resp., LM or $L \cap M$).

For $x \in P$ put $x^\perp = \{y \in P \parallel y \sim x\}$, and note that $x \in x^\perp$. The trace of a pair $\{x, y\}$ of distinct points is defined to be the set $x^\perp \cap y^\perp$ and is denoted $\operatorname{tr}\{x, y\}$ or $\{x, y\}^\perp$. We have $|\{x, y\}^\perp| = s + 1$ or $t + 1$ according as $x \sim y$ or $x \not\sim y$. More generally, if $A \subset P$, A "perp" is defined by $A^\perp = \bigcap \{x^\perp \parallel x \in A\}$. For $x \neq y$, the span of the pair $\{x, y\}$ is $\operatorname{sp}\{x, y\} = \{x, y\}^{\perp\perp} = \{u \in P \parallel u \in z^\perp \ \forall z \in x^\perp \cap y^\perp\}$. We have $|\{x, y\}^{\perp\perp}| = s + 1$ or $|\{x, y\}^{\perp\perp}| \leqslant t + 1$ according as $x \sim y$ or $x \not\sim y$.

A triad (of points) is a triple of pairwise noncollinear points. Given a triad $T = \{x, y, z\}$, a center of T is just a point of $T^\perp$.

For definitions, notations, and results neither given herein nor cited explicitly, we refer to the monograph by Payne and Thas [11].

1.2. *Restrictions on the Parameters*

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order (s, t). If $|P| = v$ and $|B| = b$, then $v = (s+1)(st+1)$ and $b = (t+1)(st+1)$; see [11, 1.2.1]. Also, $s + t$ divides $st(s+1)(t+1)$; see [11, 1.2.2].

If $s > 1$ and $t > 1$, then $t \leqslant s^2$, and dually $s \leqslant t^2$; see [11, 1.2.3]. Also, for $s > 1$ and $t > 1$, $s^2 = t$ iff each triad (of points) has a constant number of centers, in which case this constant number of centers is $s + 1$; see [11, 1.2.4].

2. Regularity, Antiregularity, 3-Regularity, and Property (G)

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order (s, t).

2.1. *Regularity*

If $x \sim y$, $x \neq y$, or if $x \not\sim y$ and $|\{x, y\}^{\perp\perp}| = t + 1$, we say that the pair $\{x, y\}$ is regular. The point x is regular provided $\{x, y\}$ is regular for all

$y \in P$, $y \neq x$. A point x is coregular provided each line incident with x is regular.

If $\mathscr{S}$ contains a regular pair $\{x, y\}$, then either $s=1$ or $s \geqslant t$; see [11, 1.3.6]. A coregular point in a GQ of order s, $s>1$, is regular iff s is even; see [11, 1.5.2].

Let x be a regular point of $\mathscr{S}$. Then the incidence structure with pointset $x^{\perp} - \{x\}$, with lineset the set of spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^{\perp} - \{x\}$, $y \not\sim z$, and with the natural incidence, is the dual of a net (cf. [2]) of order s and degree $t+1$; see [11, 1.3.1]. If in particular $s=t>1$, there arises a dual affine plane of order s. Also, in this case the incidence structure π_x with pointset $x^{\perp}$, with lineset the set of spans $\{y, z\}^{\perp\perp}$, where $y, z \in x^{\perp}$, $y \neq z$, and with the natural incidence, is a projective plane of order s.

EXAMPLES. (a) Let $Q(n, q)$ be the GQ arising from the nonsingular quadric Q of Witt index 2 in $PG(n, q)$, $n \in \{3, 4, 5\}$. All lines of $Q(n, q)$ are regular [11, 3.3.1]. All points of $Q(3, q)$ are regular; all points of $Q(4, q)$ are regular iff q is even; and no point of $Q(5, q)$ is regular [11, 3.3.1].

(b) Let $H(n, q^2)$ be the GQ arising from the nonsingular Hermitian variety H of $PG(n, q^2)$, $n \in \{3, 4\}$. All points of $H(3, q^2)$ are regular and no line of $H(3, q^2)$ is regular [11, 3.3.1]. No point and no line of $H(4, q^2)$ is regular [11, 3.3.1].

(c) Since the GQ $W(q)$ arising from a sympletic polarity ζ of $PG(3, q)$ is isomorphic to the dual of $Q(4, q)$ (cf. [11, 3.2.1]), all points of $W(q)$ are regular for all q and all lines of $W(q)$ are regular iff q is even.

(d) Let $T_2(O)$ be the GQ arising from an oval O of $PG(2, q)$. Then the point (∞) is coregular for $T_2(O)$ [11, 3.3.2].

Let $T_3(O)$ be the GQ arising from an ovoid O of $P(3, q)$. Then the point (∞) is coregular for $T_3(O)$ [11, 3.3.2].

Other results on regularity in $T_n(O)$, $n \in \{2, 3\}$, can be found in [11, 3.3.3].

(e) For any translation generalized quadrangle $\mathscr{S}^{(p)}$ the base point p is coregular [11, 8.1 and 8.2].

(f) Let $\mathscr{S}(F)$ be the GQ of order (q^2, q) arising from the flock F of the quadratic cone K of $PG(3, q)$; see [16]. Then it is easy to check that the point (∞) is regular.

THEOREM 2.1.1. *A GQ $\mathscr{S}$ of order s $(s>1)$ is isomorphic to $W(s)$ iff all its points are regular.*

Proof. See [11, 5.2.1]. ∎

2.2. *Antiregularity*

The pair $\{x, y\}$, $x \not\sim y$, is antiregular provided $|z^\perp \cap \{x, y\}^\perp| \leqslant 2$ for all $z \in P - \{x, y\}$. A point x is antiregular provided $\{x, y\}$ is antiregular for all $y \in P - x^\perp$.

If $\{x, y\}$ is antiregular with $s = t$, then s is odd; see [11, 1.5.1]. If $1 < s < t$, then $\mathcal{S}$ has no antiregular pair of points; see [11, 1.3.6]. A coregular point in a GQ of order s, $s > 1$, is antiregular iff s is odd; see [11, 1.5.2].

Let x be an antiregular point of the GQ $\mathcal{S} = (P, B, \mathrm{I})$ of order s, $s \neq 1$. Then the incidence structure with pointset $x^\perp - \{x\}$, with lineset the set of spans $\{x, y\}^{\perp\perp}$, $y \sim x$, and $y \neq x$, with circleset the set of traces $\{x, z\}^\perp$, $z \not\sim x$, and with the natural incidence, is a Laguerre plane $\mathcal{L}$ of order s; see [10]. For each point u of $\mathcal{L}$ the derived or internal affine plane of $\mathcal{L}$ at u will be denoted by $\mathcal{L}_u$ or $\pi(x, u)$; points of $\pi(x, u)$ are the points of $x^\perp$ not collinear with u, and the lines of $\pi(x, u)$ are the spans $\{x, y\}^{\perp\perp}$, with $y \sim x$ and $y \not\sim u$, and the traces $\{x, z\}^\perp$, with $z \not\sim x$ and $z \sim u$.

Examples [11, 3.3.1]. All points of $Q(4, q)$ are antiregular iff q is odd; no line of $Q(4, q)$ is antiregular. Since $W(q)$ is isomorphic to the dual of $Q(4, q)$, all lines of $W(q)$ are antiregular iff q is odd, and no point of $W(q)$ is antiregular.

Theorem 2.2.1. *Let $\mathcal{S}$ be a GQ of order s, $s \neq 1$, having an antiregular point x. Then $\mathcal{S}$ is isomorphic to $Q(4, s)$ iff there is a point y, $y \in x^\perp - \{x\}$, for which the associated affine plane $\pi(x, y)$ is Desarguesian.*

Proof. See [11, 5.2.7]. ∎

2.3. 3-*Regularity*

Let $s^2 = t > 1$, so that by 1.2 we have $|\{x, y, z\}^\perp| = s + 1$ for any triad $\{x, y, z\}$. Evidently $|\{x, y, z\}^{\perp\perp}| \leqslant s + 1$. We say $\{x, y, z\}$ is 3-regular provided $|\{x, y, z\}^{\perp\perp}| = s + 1$. The point x is called 3-regular iff each triad containing x is 3-regular.

Let $\mathcal{S}$ be a GQ of order (s, s^2), $s \neq 1$, and suppose that any triad contained in $\{x, y\}^\perp$, $x \not\sim y$, is 3-regular. Then the incidence structure with pointset $\{x, y\}^\perp$, with circleset the set of elements $\{z, z', z''\}^{\perp\perp}$, where z, z', z'' are distinct points in $\{x, y\}^\perp$ and, with the natural incidence, is an inversive plane of order s; see [11, 1.3.3].

If $\{x, y, z\} = T$ is a 3-regular triad of the GQ $\mathcal{S} = (P, B, \mathrm{I})$ of order (s, s^2), $s \neq 1$, then each point of $P - (T^\perp \cup T^{\perp\perp})$ is collinear with precisely two points of $T^\perp \cup T^{\perp\perp}$; see [11, 1.4.1].

Let $\{x, y, z\} = T$ be a 3-regular triad of the GQ $\mathcal{S} = (P, B, \mathrm{I})$ of order (s, s^2), $s > 1$, and let P' be the set of all points incident with lines of the form uv, $u \in T^{\perp}$, and $v \in T^{\perp\perp}$. If L is a line which is incident with no point of $T^{\perp} \cup T^{\perp\perp}$ and if k is the number of points in P' which are incident with L, then $k \in \{0, 2\}$ if s is odd and $k \in \{1, s+1\}$ if s is even; see [11, 2.6.1]. If s is even, if B' is the set of all lines in B which are incident with at least two points of P', and if I' is the restriction of I to $(P' \times B') \cup (B' \times P')$, then $\mathcal{S}' = (P', B', \mathrm{I}')$ is a subquadrangle of $\mathcal{S}$ of order s; moreover $\{x, y\}$ is a regular pair of $\mathcal{S}'$, with $\{x, y\}^{\perp'} = \{x, y, z\}^{\perp}$ and $\{x, y\}^{\perp'\perp'} = \{x, y, z\}^{\perp\perp}$ (see [11, 2.6.2]).

EXAMPLES. (a) All points of $Q(5, q)$ are 3-regular [11, 3.3.1]. Since $H(3, q^2)$ is isomorphic to the dual of $Q(5, q)$, all lines of $H(3, q^2)$ are 3-regular.

(b) The point (∞) of the GQ $T_3(O)$ is 3-regular [11, 3.3.2]. If $T_3(O)$ has a 3-regular point other than (∞), then O is an elliptic quadric and $T_3(O)$ is isomorphic to $Q(5, q)$ [11, 3.3.3].

THEOREM 2.3.1. (i) *A GQ of order (s, s^2), $s > 1$, is isomorphic to $T_3(O)$ iff it has a 3-regular point.*

(ii) *Let $\mathcal{S}$ be a GQ of order (s, s^2), $s > 1$, with s odd. Then $\mathcal{S} \cong Q(5, s)$ iff $\mathcal{S}$ has a 3-regular point.*

(iii) *Let $\mathcal{S}$ be a GQ of order (s, s^2) with s even. Then $\mathcal{S} \cong Q(5, s)$ iff one of the following holds:*

(a) *All points of $\mathcal{S}$ are 3-regular.*

(b) *$\mathcal{S}$ has at least one 3-regular point not incident with some regular line.*

Proof. See [11, 5.3.1 and 5.3.3]. ∎

2.4. *Property (G) for GQ of Order (s, s^2)*

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order (s, s^2), $s \neq 1$. Let x_1, y_1 be distinct collinear points. We say that the pair $\{x_1, y_1\}$ has property (G) or that $\mathcal{S}$ has property (G) at $\{x_1, y_1\}$, if every triad $\{x_1, x_2, x_3\}$, with $y_1 \in \{x_1, x_2, x_3\}^{\perp}$, is 3-regular. Then also every triad $\{y_1, y_2, y_3\}$, with $x_1 \in \{y_1, y_2, y_3\}^{\perp}$, is 3-regular. The GQ $\mathcal{S}$ has property (G) at the line L, or the line L has property (G), if each pair of points $\{x, y\}$, $x \neq y$, and $x \mathrm{I} L \mathrm{I} y$, has property (G). If (x, L) is a flag, that is, if $x \mathrm{I} L$, then we say that $\mathcal{S}$ has property (G) at (x, L), or that (x, L) has property (G), if every pair $\{x, y\}$, $x \neq y$, and $y \mathrm{I} L$, has property (G).

The point x of the GQ $\mathcal{S}$ is 3-regular iff each flag (x, L), $x \mathrm{I} L$, has property (G).

If s is even and if $\mathcal{S}$ contains a pair of points having property (G), then by 2.3 $\mathcal{S}$ contains subquadrangles of order s.

In [9] property (G) is defined for any GQ of order (s, t), $t > s > 1$.

EXAMPLE. If $\mathscr{S}(F)$ is the GQ of order (q^2, q) arising from the flock F of the quadratic cone K of $PG(3, q)$, then by [9] $\mathscr{S}$ has property (G) at the point (∞).

Goal of the paper. In the subsequent sections GQ of order (s, s^2), $s \neq 1$, having property (G) at some pair of points, or at some line, or at some flag, will be considered. In particular translation generalized quadrangles of order (s, s^2), $s \neq 1$, with base point (∞) and having property (G) either at some flag $((\infty), L)$ or at some line L incident with the point (∞), are studied in detail.

3. GENERALIZED QUADRANGLES OF ORDER (q, q^2), $q \neq 1$, SATISFYING PROPERTY (G)

3.1. *The Affine Space $\mathscr{S}_{xy}$ (S. E. Payne and J. A. Thas)*

Let $\mathscr{S} \doteq (P, B, \mathrm{I})$ be a GQ of order (q, q^2), $q \neq 1$, for which the pair of points $\{x, y\}$, $x \sim y$, has property (G). Now we introduce the following incidence structure $\mathscr{S}_{xy} = (P_{xy}, B_{xy}, \mathrm{I}_{xy})$:

(i) $P_{xy} = x^\perp - \{x, y\}^{\perp\perp}$.

(ii) Elements of B_{xy} are of two types: (a) the sets $\{y, z, u\}^{\perp\perp} - \{y\}$, with $\{y, z, u\}$ a triad having x as center, and (b) the sets $\{x, w\}^\perp - \{x\}$, with $x \sim w \not\sim y$.

(iii) I_{xy} is containment.

Then $|P_{xy}| = q^3$, B_{xy} contains $q^3(q^3 - q)/q(q-1) = q^4 + q^3$ elements of type (a) and q^2 elements of type (b).

LEMMA 3.1.1. *Let M_1, M_2 be different elements of type* (a) *of B_{xy}, let L_1, L_2 be different elements of type* (b) *of B_{xy}, and assume that $L_i \cap M_j \neq \varnothing$ for all $i, j \in \{1, 2\}$. Then each element of type* (b) *of B_{xy} which contains a point of M_1 also contains a point of M_2.*

Proof. Let $L_i \cap M_j = \{x_{ij}\}$. Since $M_1 \neq M_2$, we have $\{x_{11}, x_{21}\} \neq \{x_{12}, x_{22}\}$. Suppose, e.g., that $x_{21} \neq x_{22}$. Let $u \in M_1$, $u \notin L_1$, $u \notin L_2$, $N = \{x, u\}^\perp - \{x\}$. If N does not contain a point of $\{y, x_{11}, x_{22}\}^{\perp\perp}$ then by 2.3, and since the point $x \in \{y, x_{11}, x_{22}\}^\perp$ is collinear with u, the point u is collinear with a second point x' of $\{y, x_{11}, x_{22}\}^\perp$. So x' is collinear with y, x_{11}, x_{22}, u. Hence $x' \in \{y, x_{11}, u\}^\perp$ and so $x' \sim x_{21}$, giving a triangle $\{x', x_{21}, x_{22}\}$, a contradiction. Consequently N contains a point of

$\{y, x_{11}, x_{22}\}^{\perp\perp}$. If $x_{11} \neq x_{12}$, then an analogous argument shows that N contains a point of M_2; if $x_{11} = x_{12}$, then, as $\{y, x_{11}, x_{22}\}^{\perp\perp} = M_2 \cup \{y\}$, it is clear that N contains a point of M_2. ▮

Now we introduce the set H_{xy}. The elements of H_{xy} are of two types:

(a) The sets $\{x, z\}^{\perp} - \{y\}$, with $x \not\sim z$ and $y \in \{x, z\}^{\perp}$, and

(b) each set which is the union of all elements of type (b) of B_{xy} containing a point of some line M of type (a) of B_{xy}.

The set H_{xy} contains q^3 elements of type (a), and by Lemma 3.1.1 it contains $(q^4 + q^3)$: $q^2(q^2 - q)/q(q-1) = q^2 + q$ elements of type (b).

LEMMA 3.1.2. *The elements of P_{xy} and B_{xy} in an element of H_{xy} are the points and lines of a $2-(q^2, q, 1)$ design, that is, an affine plane of order q.*

Proof. Easy, using property (G) and Lemma 3.1.1. ▮

THEOREM 3.1.3. *The elements of P_{xy}, B_{xy}, and H_{xy}, respectively, are the points, lines, and planes of affine space $AG(3, q)$.*

Proof. We have $|P_{xy}| = q^3$, each element of B_{xy} contains q elements of P_{xy}, and any two distinct elements of P_{xy} are contained in a unique element of B_{xy}. Hence $\mathcal{S}_{xy}$ is a $2-(q^3, q, 1)$ design.

If $q = 3$, then by [11, 5.3.2] we have $\mathcal{S} \cong Q(5, 3)$. By [11, 3.2.4] $Q(5, 3) \cong T_3(O)$, with O an elliptic quadric of $PG(3, 3)$. Now the statement of the theorem is easily checked by taking for y the point (∞) and for x any point distinct from (∞) but collinear with (∞).

Now assume $q > 3$. By a theorem of Buekenhout [1] it is sufficient to show that any three points of P_{xy}, not contained in a common element of B_{xy}, are contained in a common element of H_{xy}, Let z, u, w be three distinct points of P_{xy}, not in a common element of B_{xy}, and not in a common element of type (b) of H_{xy}. Then no two points of $\{z, u, w\}$ are collinear in $\mathcal{S}$ (that is, no two points of $\{z, u, w\}$ are incident with a common line of $\mathcal{S}$ through x), and $\{y, z, u\}^{\perp\perp}$ has no point in common with $\{x, w\}^{\perp}$. Since $w \sim x$ and $x \in \{y, z, u\}^{\perp}$, so by 2.3 w is collinear with a second point x' of $\{y, z, u\}^{\perp}$. Clearly $\{x, x'\}^{\perp}$ contains y, z, u, w, and hence z, u, w are contained in an element of type (a) of H_{xy}. ▮

COROLLARY 3.1.4. *If $\mathcal{S} = (P, B, \mathrm{I})$ is a GQ of order (q, q^2), $q \neq 1$, for which at least one pair of points $\{x, y\}$, $x \sim y$, has property (G), then q is a prime power.*

Proof. Immediate from Theorem 3.1.3. ▮

THEOREM 3.1.5. *Let $\mathcal{S}=(P, B, \mathrm{I})$ be a GQ of order (q, q^2), q even, for which at least one pair of points $\{x, y\}$, $x \sim y$, has property* (G). *By* 2.3 *each element of type* (a) *of B_{xy} defines a subquadrangle $\mathcal{S}'$ of order q of $\mathcal{S}$. In this way there arise q^3+q^2 subquadrangles of order q of $\mathcal{S}$. Also, in any such subquadrangle each pair $\{y, u\}$ with $x \in \{y, u\}^{\perp}$ (and so each pair $\{x, z\}$ with $y \in \{x, z\}^{\perp}$) is regular.*

Proof. Assume that the elements M_1, M_2 of type (a) of B_{xy} define the same subquadrangle $\mathcal{S}'$ of order q. Let $M_i \cup \{y\}=\{y, u_i, z_i\}^{\perp\perp}$, $i=1,2$. Then by 2.3 the pair $\{y, u_i\}$, $i=1,2$, is regular in $\mathcal{S}'$ and $\{y, u_i\}^{\perp'\perp'}=M_i \cup \{y\}$, $i=1,2$. Since $\{y, u_i\}^{\perp'\perp'}=\{v_i, w_i\}^{\perp'\perp'}$ for any two points $v_i, w_i \in \{y, u_i\}^{\perp'\perp'}$, $i=1,2$, we have $\{y, u_1\}^{\perp'\perp'} \cap \{y, u_2\}^{\perp'\perp'}=\{y\}$. Also, as $x \in \{y, u_i\}^{\perp'}$ with $i=1,2$, any element of type (b) of B_{xy} containing a point of M_1 also contains a point of M_2. Hence M_1 and M_2 are parallel lines in the affine plane of type (b) of $AG(3, q)$ defind by M_1.

Conversely, let M_1, M_2 be distinct parallel lines of type (a) of an affine plane of type (b) of $AG(3, q)$. Suppose that $M_i \cup \{y\}=\{y, u_i, z_i\}^{\perp\perp}$, $i=1,2$, where $u_1 \sim u_2$ and $z_1 \sim z_2$. If $w \in \{y, u_1, z_1\}^{\perp} \cap \{y, u_2, z_2\}^{\perp}$, $w \neq x$, then there arises a triangle $\{w, u_1, u_2\}$, a contradiction. Hence $\{y, u_1, z_1\}^{\perp} \cap \{y, u_2, z_2\}^{\perp}=\{x\}$. Now we show that each point r of $\{y, u_2, z_2\}^{\perp}-\{x\}$ is incident with a line of $\mathcal{S}$ joining a point of $\{y, u_1, z_1\}^{\perp\perp}$ to a point of $\{y, u_1, z_1\}^{\perp}$. Suppose the contrary. By 2.3, r is collinear with two points y, y' of $\{y, u_1, z_1\}^{\perp\perp}$. Now, let $\{y''\}=\{y, u_2, z_2\}^{\perp\perp} \cap \{x, y'\}^{\perp}$. Then there arises the triangle $\{y', y'', r\}$, a contradiction. Consequently r is incident with a line of $\mathcal{S}$ joining a point w_1 of $\{y, u_1, z_1\}^{\perp\perp}$ to a point v_1 of $\{y, u_1, z_1\}^{\perp}$. (If $w_1 \neq y$, then there arises a triangle $\{y, r, v_1\}$, a contradiction. So $w_1=y$ and the line ry of $\mathcal{S}$ contains a point v_1 of $\{y, u_1, z_1\}^{\perp}$.) Hence r is a point of the subquadrangle $\mathcal{S}'$ of order q defined by M_1, and so $\{y, u_2, z_2\}^{\perp}$ is contained in $\mathcal{S}'$. Since M_2 is a line of the affine plane of type (b) defined by M_1, also M_2 is contained in $\mathcal{S}'$. We conclude that $\mathcal{S}'$ is also the subquadrangle of order q defined by M_2.

From the preceding sections it follows that the elements of type (a) of B_{xy} define $(q^4+q^3)/q=q^3+q^2$ subquadrangles of order q of $\mathcal{S}$.

Now consider a point u of the subquadrangle $\mathcal{S}'$ defined by M_1, with $y \not\sim u$ and $x \in \{y, u\}^{\perp}$. As $\{x, u\}^{\perp}$ has a point in common with M_1, the point u belongs to the affine plane π defined by M_1. Let M_2 be the line of π parallel to M_1. Then M_2 also defines the subquadrangle $\mathcal{S}'$. It follows that $\{y, u\}$ is regular in $\mathcal{S}'$. ∎

Remark 3.1.6. Assume q even or odd. From the second section of the proof of the preceding theorem and by interchanging roles of x and y in defining affine planes of type (b), we see that M_1 and M_2 are parallel lines of type (a) in an affine plane of type (b) of $AG(3, q)$ iff $(M_1 \cup \{y\})^{\perp} - \{x\} = N_1$ and $(M_2 \cup \{y\})^{\perp} - \{x\} = N_2$ are parallel lines in the affine plane defined by y and N_1. ∎

3.2. $\mathcal{S}$ *Has Property* (G) *either at the Line L or at the Flag* (x, L)

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order (q, q^2), q even, which satisfies property (G) at the flag (x, L), with $L = xy$. Further, let x be a center of the triad $\{y, z, u\}$ and let $\mathcal{S}'$ be the subquadrangle of order q containing $\{y, z, u\}^{\perp}$ and $\{y, z, u\}^{\perp\perp}$.

THEOREM 3.2.1. *The point x is regular in $\mathcal{S}'$. If $\mathcal{S}$ satisfies property* (G) *at the line L, then all points of L are regular in $\mathcal{S}'$, L is regular in $\mathcal{S}'$, and L is regular in $\mathcal{S}$.*

Proof. Let w be a point of $\mathcal{S}'$, with $w \not\sim x$ and $w \not\sim y$. Further, let r, r', r'' be distinct points of $\{x, w\}^{\perp'}$, with $r \mathrm{I} L$. Then $\{r, r', r''\}^{\perp}$ and $\{r, r', r''\}^{\perp\perp}$ are contained in a subquadrangle $\mathcal{S}''$ of order q of $\mathcal{S}$. Clearly the lines xr, xr', xr'', wr, wr', wr'' are common lines of $\mathcal{S}'$ and $\mathcal{S}''$. Then by [11, 2.3.1] the common points and lines of $\mathcal{S}'$ and $\mathcal{S}''$ form a subquadrangle $\mathcal{S}' \cap \mathcal{S}''$ of order (q, t'), $t' > 1$, of $\mathcal{S}''$. Now by [11, 2.2.2] we have $\mathcal{S}' \cap \mathcal{S}'' = \mathcal{S}'$, and so $\mathcal{S}' = \mathcal{S}''$. It follows that $\{r, r'\}^{\perp'\perp'} = \{r, r', r''\}^{\perp\perp}$, $\{r, r'\}^{\perp'} = \{r, r', r''\}^{\perp}$, and so $\{r, r'\}$ and $\{x, w\}$ are regular in $\mathcal{S}'$. Hence x is regular in $\mathcal{S}'$.

Assume now that $\mathcal{S}$ satisfies property (G) at the line L. With the notations of the previous section and interchanging the roles of x and y, we see that also y is regular in $\mathcal{S}'$. Since y and r play the same role in $\mathcal{S}'$, also r is regular in $\mathcal{S}'$. It follows immediately that all points of L are regular in $\mathcal{S}'$. Hence also the line L is regular in $\mathcal{S}'$(cf. 2.1). Consider now a line M of $\mathcal{S}$, $M \not\sim L$. Let $y \sim x' \mathrm{I} M$, $x \sim y' \mathrm{I} M$, and $x'' \in \{y, y'\}^{\perp} - \{x, x'\}$. The subquadrangle $\bar{\mathcal{S}}$- of order q defined by x, x', x'' contains L and M. Since L is regular in $\mathcal{S}$, the span $\{L, M\}^{\perp'\perp'} = \{L, M\}^{\perp\perp}$ has size $q+1$. We conclude that L is regular in $\mathcal{S}$. ∎

From now on let q be odd or even and let $\mathcal{S}$ satisfy property (G) at the flag (x, L). Let R be $\{u, u_1, u_2\}^{\perp\perp} = \{u, u_1, u_2, ..., u_q\}$, with $\{u, u_1, u_2\}$ a triad having x as center and with $u \mathrm{I} L$, and let L_i be the line incident with x and the point u_i, $i = 1, 2, ..., q$. Further, let P_{xR} be the set of all points, different from x, collinear with x and a point of R, let $B_{xR} = \{L, L_1, L_2, ..., L_q\}$, and let C_{xR} be the set having as elements the sets $\{w, w_1, w_2\}^{\perp\perp}$ with $w \mathrm{I} L$, $w_1 \mathrm{I} L_1$, $w_2 \mathrm{I} L_2$, and $x \notin \{w, w_1, w_2\}$. Also, let incidence I_{xR} between elements of P_{xR} and elements of B_{xR} be induced by the incidence in $\mathcal{S}$, and let incidence I_{xR} between elements of P_{xR} and elements of C_{xR} be containment.

THEOREM 3.2.2. *The incidence structure $\mathscr{L}=(P_{xR}, B_{xR}, C_{xR}, \mathrm{I}_{xR})$, with pointset P_{xR}, lineset B_{xR}, circleset C_{xR}, and incidence I_{xR}, is a Laguerre plane of order q. Also, for each point $y \mathrm{I} L$, $y \neq x$, the derived or internal affine plane $\mathscr{L}_y$ of $\mathscr{L}$ at y is the affine plane $AG(2, q)$; hence, for q odd $\mathscr{L}$ is the classical Laguerre plane, that is, arises from the quadratic cone in $PG(3, q)$.*

Proof. Clearly $|P_{xR}|=q^2+q$, $|B_{xR}|=q+1$, $|C_{xR}|=q^3$, each line of $\mathscr{L}$ is incident with q points of $\mathscr{L}$, and each circle of $\mathscr{L}$ is incident with $q+1$ points of $\mathscr{L}$.

Let $\{w, w_1, w_2\}^{\perp\perp}$ and $\{w', w_1, w_2\}^{\perp\perp}$ be elements of C_{xR}, $w \mathrm{I} L$, $w' \mathrm{I} L$, $w \neq w'$, $w_1 \mathrm{I} L_1$, $w_2 \mathrm{I} L_2$. Consider $w'' \in \{w, w_1, w_2\}^{\perp\perp} - \{w, w_1, w_2\}$ and assume $\{x, w''\}^{\perp}$ has no point in common with $\{w', w_1, w_2\}^{\perp\perp}$. As there are no triangles in $\mathscr{S}$, w'' is collinear with no point of $\{w', w_1, w_2\}^{\perp\perp}$. Hence w'' is collinear with two distinct points x, r of $\{w', w_1, w_2\}^{\perp}$. Since $r \sim w''$, $r \sim w_1$, and $r \sim w_2$, we have $r \sim w$. So there arises a triangle with vertices w, w', r, a contradiction. Consequently $\{x, w''\}^{\perp}$ has a point in common with $\{w', w_1, w_2\}^{\perp\perp}$.

Consider again $\{w, w_1, w_2\}^{\perp\perp}$. We show that $\{w, w_1, w_2\}^{\perp\perp}$ has a point in common with each of $L, L_1, L_2, ..., L_q$. If $w=u$ we are done by Lemma 3.1.1. If $w \neq u$, then consider $\{u, w_1, w_2\}^{\perp\perp}$, and apply the previous case to $\{u, u_1, u_2\}^{\perp\perp}$ and $\{u, w_1, w_2\}^{\perp\perp}$ and the first section of this proof to $\{u, w_1, w_2\}^{\perp\perp}$ and $\{w, w_1, w_2\}^{\perp\perp}$.

It follows that each line of $\mathscr{L}$ and each circle of $\mathscr{L}$ have exactly one point in common.

Since $\mathscr{S}$ satisfies property (G) at (x, L), each circle of $\mathscr{L}$ is uniquely defined by any three of its points; hence two distinct circles of $\mathscr{L}$ have at most two points in common. Since each triad of points of $\mathscr{L}$ is on at most one circle of $\mathscr{L}$, there are at most $[(q^2+q)\,q^2(q^2-q)]/[(q+1)\,q(q-1)]=q^3$ circles. As there are exactly q^3 circles, we conclude that each triad of points of $\mathscr{L}$ is on exactly one circle of $\mathscr{L}$.

Consequently $\mathscr{L}$ is a Laguerre plane of order q.

Let $y \mathrm{I} L$, $y \neq x$. Then the derived or internal affine plane $\mathscr{L}_y$ of $\mathscr{L}$ at y is the affine plane in the statement of Lemma 3.1.2. Now by Theorem 3.1.3 this affine plane $\mathscr{L}_y$ is $AG(2, q)$. Finally, if q is odd, then by [10] $\mathscr{L}$ is the classical Laguerre plane, that is, the Laguerre plane arising from the quadratic cone of $PG(3, q)$. ∎

Let O be an oval (that is, a set of $q+1$ points no three of which are collinear) in a plane $PG(2, q)$ of some $PG(3, q)$, and let x be a point of $PG(3, q)$ not contained in $PG(2, q)$. Further, let K be the cone projecting O from x. Then the points of $K-\{x\}$, the lines on K, and the intersections of K by planes not containing x, are the points, lines, and circles of a Laguerre plane denoted $\mathscr{L}(O)$ (the incidence is the natural one). If q is odd, then O is necessarily an irreducible conic (see 8.2.4 of [4]) and so $\mathscr{L}(O)$ is the Laguerre plane arising from the quadratic cone; that is, $\mathscr{L}(O)$ is the classical Laguerre plane. Further, each known Laguerre plane is isomorphic to a $\mathscr{L}(O)$, for some oval O of some $PG(2, q)$.

THEOREM 3.2.3. *Suppose that q is even and that the Laguerre plane $\mathscr{L}$ of Theorem 3.2.2 is isomorphic to some $\mathscr{L}(O)$. Let C be a circle of $\mathscr{L}$ and let $\mathscr{S}'$ be the subquadrangle of order q containing C and $C^{\perp}$. Then the projective plane π_x defined by the regular point x of $\mathscr{S}'$ is Desarguesian.*

Proof. So we assume that q is even and that $\mathscr{L} \cong \mathscr{L}(O)$ for some oval O of some plane $PG(2, q)$ of $PG(3, q)$. The lines of π_x not containing x are circles of $\mathscr{L}$ (cf. 2.1 and 3.2). Any two of these lines have exactly one point in common, so we have q^2 circles of $\mathscr{L}$ which are mutually tangent. Let $C_1, C_2, ..., C_{q^2}$ be the corresponding circles of $\mathscr{L}(O)$. Since q is even, the $q+1$ tangent planes of the cone K defining $\mathscr{L}(O)$ contain a common line, the nucleus line N of K. The nucleus of the oval C_i is the intersection of N with any tangent line of C_i. Since any two ovals of $\{C_1, C_2, ..., C_{q^2}\}$ are mutually tangent, they have a common tangent line, hence have the same nucleus. Consequently $C_1, C_2, ..., C_{q^2}$ have a common nucleus $n \in N$. Projecting K minus its vertex $\bar{x}$ from n, we see that the incidence structure formed by the points of $K - \{\bar{x}\}$ and the circles $C_1, C_2, ..., C_{q^2}$ is isomorphic to the dual affine plane with as points the lines of $PG(3, q)$ containing n and a point of $K - \{\bar{x}\}$, and with as lines the planes of $PG(3, q)$ containing n but not $\bar{x}$. Clearly this dual affine plane is Desarguesian. Hence the dual affine plane obtained from the projective plane π_x by deleting x and the lines through x is Desarguesian. We conclude that the projective plane π_x is Desarguesian. ∎

4. TRANSLATION GENERALIZED QUADRANGLES OF ORDER (s, s^2), $s \neq 1$, SATISFYING PROPERTY (G)

4.1. *GQ Arising from Flocks*

Let F be a flock of the quadratic cone K with vertex x of $PG(3, q)$, that is, a partition of $K - \{x\}$ into q disjoint irreducible conics. Then, by [16], with F there corresponds a GQ $\mathscr{S}(F)$ of order (q^2, q). In [9] it was shown that $\mathscr{S}(F)$ satisfies property (G) at its point (∞). If $\mathscr{S}(F)$ satisfies property (G) at some point $z \in (\infty)^{\perp} - \{(\infty)\}$, then by [12], $\mathscr{S}(F)$ is isomorphic to the classical GQ $H(3, q^2)$ arising from the nonsingular Hermitian variety H of $PG(3, q^2)$. Also, $\mathscr{S}(F) \cong H(3, q^2)$ iff all the planes of the conics of F contain a common line of $PG(3, q)$; see [16]. Using the fact that any $\mathscr{S}(F)$ is an elation generalized quadrangle (EGQ) (that is, admits a group G

(elation group) of collineations, each fixing the point (∞) linewise, which acts regularly on the points not collinear with (∞)) it is then clear that for any nonclassical $\mathcal{S}(F)$ the point (∞) is fixed by any collineation and that a nonclassical GQ of order (q^2, q) is a $\mathcal{S}(F)$ for at most one point (∞).

Now suppose that the nonclassical GQ $\mathcal{S}(F)$ of order (s, t), $s = t^2$, is a translation generalized quadrangle (TGQ) with base line L; that is, assume that:

(i) The GQ $\mathcal{S}(F)$ admits a group G of collineations, each fixing L pointwise, which acts regularly on the set of lines not concurrent with L (that is, $\mathcal{S}(F)$ is an EGQ with base line L and elation group G).

(ii) The group G is abelian.

In such a case, $G - \{1\}$ is the set of all collineations of $\mathcal{S}(F)$ fixing L pointwise and having no fixed line M with $M \not\sim L$ (see [11, 8.6.4]). The group G is called the translation group of the TGQ with base line L. As $\mathcal{S}(F)$ is not classical, the point (∞) is fixed by all elements of G. Consequently (∞) is incident with L. Also, N. L. Johnson [6] proved that for such an $\mathcal{S}(F)$, s (or t) is necessarily odd.

By [11, Section 8.7], the dual $\widehat{\mathcal{S}(F)}$ of the GQ $\mathcal{S}(F)$ is then isomorphic to a GQ of type $T(n, 2n, q) = T(O)$. A GQ $T(n, 2n, q)$ is described as follows.

In $PG(4n-1, q)$ consider a set $O(n, 2n, q) = O$ of $q^{2n}+1$ $(n-1)$-dimensional subspaces $(\infty) = PG^{(0)}(n-1, q), PG^{(1)}(n-1, q), \ldots, PG^{(q^{2n})}(n-1, q)$, every three of which generate a $PG(3n-1, q)$, and such that each element $PG^{(i)}(n-1, q)$ of O is contained in a $PG^{(i)}(3n-1, q)$ having no point in common with any $PG^{(j)}(n-1, q)$, $j \neq i$ and $i = 0, 1, \ldots, q^{2n}$. It is easy to check that $PG^{(i)}(3n-1, q)$ is uniquely determined, $i = 0, 1, \ldots, q^{2n}$. The space $PG^{(i)}(3n-1, q)$ is called the tangent space of O at $PG^{(i)}(n-1, q)$. Embed $PG(4n-1, q)$ in a $PG(4n, q)$, and define $T(n, 2n, q) = T(O)$ as follows. Points are of three types:

(i) The points of $PG(4n, q) - PG(4n-1, q)$.

(ii) The $3n$-dimensional subspaces of $PG(4n, q)$ which intersect $PG(4n-1, q)$ in one of the $PG^{(i)}(3n-1, q)$.

(iii) The symbol (L).

Lines are of two types:

(a) The n-dimensional subspaces of $PG(4n, q)$ which intersect $PG(4n-1, q)$ in a $PG^{(i)}(n-1, q)$.

(b) The elements of O.

Incidence in $T(O)$ is defined as follows: A point of type (i) is incident only with lines of type (a); here the incidence is that of $PG(4n, q)$. A point of type (ii) is incident with all lines of type (a) contained in it and with the unique element of O contained in it. The point (L) is incident with no line of type (a) and with all lines of type (b). This GQ $T(O)$ has order (q^n, q^{2n}).

Here the base point (∞) of the EGQ $\mathcal{S}(F)$ corresponds to the line $PG^{(o)}(n-1, q) = (\infty)$ of type (b) of $T(O)$, and the base line L of the TGQ $\mathcal{S}(F)$ corresponds to the point (L) of type (iii).

Consider any GQ $T(n, 2n, q) = T(O)$. By [11, 8.7.2] the $q^{2n}+1$ tangent spaces of $O = O(n, 2n, q)$ form an $O^* = O^*(n, 2n, q)$ in the dual space of $PG(4n-1, q)$. So in addition to $T(O)$ there arises a TGQ $T(O^*)$, also denoted $T^*(O)$, with the same parameters. The TGQ $T^*(O)$ is called the translation dual of the TGQ $T(O)$.

Each TGQ $\mathcal{S}$ of order (s, s^2), $s \neq 1$, with base point x has a kernel $\mathcal{K}$, which is a field with a multiplicative group isomorphic to the group of all collineations of $\mathcal{S}$ fixing the point x, and any given point not collinear with x, linewise; we have $|\mathcal{K}| \leqslant s$ (see [11]). The field $GF(q)$ is a subfield of $\mathcal{K}$ iff $\mathcal{S}$ is of type $T(n, 2n, q)$ (here $s = q^n$); see [11]. The TGQ $\mathcal{S}$ is isomorphic to a $T_3(O)$ of Tits, O an ovoid of $PG(3, s)$, iff $|\mathcal{K}| = s$.

Finally, the TGQ $T(O)$ of order (q^n, q^{2n}) and its translation dual $T(O^*)$ have isomorphic kernels.

4.2. *The Known GQ $\mathcal{S}(F)$ which Are also TGQ*

If $\mathcal{S}(F)$ is the classical GQ $H(3, q^2)$, then it is a TGQ with base line L, L any line of $\mathcal{S}(F)$. The dual $\widehat{\mathcal{S}(F)}$ of $\mathcal{S}(F)$ is isomorphic to $T_3(O)$, O an elliptic quadric of $PG(3, q)$. Hence the kernel $\mathcal{K}$ is the field $GF(q)$. Also, $\widehat{\mathcal{S}(F)}$ is isomorphic to its translation dual $\widehat{\mathcal{S}(F)}^*$.

Let K be the quadratic cone with equation $X_0X_1 = X_2^2$ of $PG(3, q)$, q odd. Then the q planes π_t with equation $tX_0 - mt^\sigma X_1 + X_3 = 0$, $t \in GF(q)$, m a given nonsquare of $GF(q)$, and σ a given automorphism of $GF(q)$, define a flock F of K; see [16]. All the planes π_t contain the exterior point $(0, 0, 1, 0)$ of K. This flock F is linear; that is, all the planes π_t contain a common line, iff $\sigma = 1$. Conversely, every nonlinear flock F of K for which the planes of the q conics all contain a common point, is of the type just described; see [16]. The corresponding GQ $\mathcal{S}(F)$ were first discovered by W. M. Kantor [8]. Any such GQ $\mathcal{S}(F)$ is a TGQ for some base line, and so the dual $\widehat{\mathcal{S}(F)}$ of $\mathcal{S}(F)$ is isomorphic to some $T(O)$. The kernel $\mathcal{K}$ is the fixed field of σ; see [13]. Also, in [9] it is proved that $T(O)$ is isomorphic to its translation dual $T(O^*)$.

Let K be again the quadratic cone with equation $X_0X_1 = X_2^2$, $q = 3^r$, and $r > 2$. Then the q planes π_t with equation $tX_0 - (mt + m^{-1}t^9)X_1 + t^3X_2 + X_3 = 0$, $t \in GF(q)$, m a given nonsquare of $GF(q)$, define a flock F of K; see [9]. The corresponding GQ $\mathcal{S}(F)$ is a TGQ for some base line, and so the dual $\widehat{\mathcal{S}(F)}$ of $\mathcal{S}(F)$ is isomorphic to some $T(O)$. By [13], the kernel $\mathcal{K}$ is here $GF(3)$. Payne [9] proves that $T(O)$ is not isomorphic to its translation dual $T(O^*)$. Also, he shows that $T(O^*)$ is an EGQ which does not arise from a flock. These GQ were called by Payne the Roman GQ.

4.3. *TGQ of Order* (s, s^2), $s \neq 1$, *Satisfying Property* (*G*)

Let $\mathcal{S} = (P, B, \mathrm{I})$ be a GQ of order (s, s^2), $s \neq 1$, having property (G) at a pair of points $\{x, y\}$, $x \mathrm{I} L \mathrm{I} y$. Further assume that $\mathcal{S}$ is a TGQ with base point x, $x \mathrm{I} L$. As the collineation group of $\mathcal{S}$ is transitive on the points of L different from x, it follows that $\mathcal{S}$ satisfies property (G) at the flag (x, L). The GQ $\mathcal{S}$ is isomorphic to a GQ $T(O)$, $O = \{\zeta, \zeta_1, ..., \zeta_{q^{2n}}\}$ with $q^n = s$. The point x corresponds to the point (∞) of type (iii) of $T(O)$, and we assume that L corresponds to the line ζ of type (b) of $T(O)$. The tangent spaces of O at the elements $\zeta, \zeta_1, ...,$ are respectively denoted by $\tau, \tau_1, ...$

THEOREM 4.3.1. *A GQ $T(O)$ satisfies property (G) at $\{(\infty), \bar{\zeta}\}$ [or, equivalently, at the flag $((\infty), \zeta)$], with $\bar{\zeta}$ a point of type* (ii) *incident with the line ζ of type* (b) *iff for any two elements ζ_i, ζ_j $(i \neq j)$ of $O - \{\zeta\}$ the $(n-1)$-dimensional space $PG(n-1, q) = \tau \cap \tau_i \cap \tau_j$ of $PG(4n-1, q)$ is contained in exactly $q^n + 1$ tangent spaces of O.*

Proof. Suppose that $T(O)$ satisfies property (G) at $((\infty), \zeta)$. Let ζ_i, ζ_j be distinct elements of $O - \{\zeta\}$ and put $PG(n-1, q) = \tau \cap \tau_i \cap \tau_j$. By [11, Section 8.7] each point of $PG(n-1, q)$ is contained in exactly $q^n + 1$ tangent spaces of O. Let $u \in PG(n-1, q)$, let N be a line of $PG(4n, q)$ through u not contained in $PG(4n-1, q)$, and let u_1, u_2 be distinct points of N not contained in $PG(4n-1, q)$. Then each of the points $N\tau$, $N\tau_i$, $N\tau_j$ of type (ii) of $T(O)$ is collinear with each of the points (∞), u_1, u_2 of $T(O)$. Then $\{(\infty), u_1, u_2\}^\perp$ is the set $\{N\tau, N\tau_i, N\tau_j, ...\}$ with $\tau, \tau_i, \tau_j, ...$ the $q^n + 1$ tangent spaces of O through u. As $T(O)$ satisfies property (G) at ζ, we have $|\{(\infty), u_1, u_2\}^{\perp\perp}| = q^n + 1$. Hence $N\tau \cap N\tau_i \cap N\tau_j \cap \cdots = \eta$ contains q^n points of $PG(4n, q) - PG(4n-1, q)$. Consequently η is an n-dimensional space. That is, $\tau \cap \tau_i \cap \tau_j \cap \cdots$ is $(n-1)$-dimensional; that is, $\tau \cap \tau_i \cap \tau_j \cap \cdots$ is $PG(n-1, q)$. We conclude that $PG(n-1, q)$ is contained in exactly $q^n + 1$ tangent spaces of O.

Conversely, let ζ_i, ζ_j be any two distinct elements of $O - \{\zeta\}$, put $PG(n-1, q) = \tau \cap \tau_1 \cap \tau_2$, and assume that $PG(n-1, q)$ is always contained in exactly $q^n + 1$ tangent spaces of O. Let $\bar{\zeta}$, $\bar{\zeta}_i$, $\bar{\zeta}_j$ be distinct points of type (ii) collinear with (∞), where $\zeta \subset \bar{\zeta}$, $\zeta_i \subset \bar{\zeta}_i$, $\zeta_j \subset \bar{\zeta}_j$. Then $\bar{\zeta} \cap \bar{\zeta}_i \cap \bar{\zeta}_j$ is an n-dimensional space $PG(n, q)$. The $(n-1)$-dimensional space $PG(n-1, q) = PG(n, q) \cap PG(4n-1, q) = \tau \cap \tau_i \cap \tau_j$ is contained in exactly $q^n + 1$ tangent spaces $\tau, \tau_i, \tau_j, ...$ of O. The $q^n + 1$ spaces $PG(n, q)\tau = \bar{\zeta}$, $PG(n, q)\,\tau_i = \bar{\zeta}_i$, $PG(n, q)\,\tau_j = \bar{\zeta}_j, ...$ are points of type (ii) of $T(O)$, each of which is collinear with (∞) and with the q^n points of type (i) in $PG(n, q)$.

Hence the triad $\{\bar{\zeta}, \bar{\zeta}_i, \bar{\zeta}_j\}$ is 3-regular. We conclude that $\mathcal{S}$ satisfies property (G) at $((\infty), \zeta)$. ▮

THEOREM 4.3.2. *If q is even and if the TGQ $T(O)$ satisfies property (G) at the flag $((\infty), \zeta)$, then the translation dual $T(O^*)$ satisfies property (G) at the flag $((\infty), \tau)$, with τ the tangent space of O at ζ.*

Proof. Suppose that $T(O)$ satisfies property (G) at the flag $((\infty), \zeta)$, with q even. Let $\bar{\zeta}, \bar{\zeta}_i, \bar{\zeta}_j$ be distinct points of type (ii) collinear with (∞), where $\zeta \subset \bar{\zeta}, \zeta_i \subset \bar{\zeta}_i, \zeta_j \subset \bar{\zeta}_j$. Then $\{\bar{\zeta}, \bar{\zeta}_i, \bar{\zeta}_j\}^{\perp}$ and $\{\bar{\zeta}, \bar{\zeta}_i, \bar{\zeta}_j\}^{\perp\perp}$ are contained in a subquadrangle $\mathcal{S}'$ of order q^n of $T(O)$. The space $\bar{\zeta} \cap \bar{\zeta}_i \cap \bar{\zeta}_j$ is an n-dimensional space $PG(n, q)$, and the space $PG(n-1, q) = PG(n, q) \cap PG(4n-1, q) = \tau \cap \tau_i \cap \tau_j$ is contained in exactly $q^n + 1$ tangent spaces $\tau, \tau_i, \tau_j, \ldots$ of O. Then the points of $PG(4n, q) - PG(4n-1, q)$ in $PG(n, q)\zeta$, $PG(n, q)\,\zeta_i$, $PG(n, q)\,\zeta_j, \ldots$ are the points of type (i) in $\mathcal{S}'$. Now we consider a line of type (a) concurrent with ζ and incident with a point of type (i) in the space $PG(n, q)\,\zeta_i$. As this line is also a line of $\mathcal{S}'$, it is incident with a point of type (i) in each of the spaces $PG(n, q)\,\zeta_j, \ldots$. Hence the spaces $PG(n, q)\,\zeta_j, \ldots$ are contained in the $3n$-dimensional space $PG(n, q)\,\zeta\zeta_i = PG(3n, q)$. Consequently the spaces $PG(n-1, q), \zeta, \zeta_i, \zeta_j, \ldots$ are contained in a common $PG(3n-1, q) = PG(3n, q) \cap PG(4n-1, q)$. So, by [11, 8.7.2], the $(3n-1)$-dimensional space $\zeta\zeta_i\zeta_j$ contains exactly $q^n + 1$ elements of O. Applying Theorem 4.3.1 to O^* now gives that the translation dual $T(O^*)$ of $T(O)$ has property (G) at the flag $((\infty), \tau)$. ▮

Remark 4.3.3. From the proof of Theorem 4.3.2 follows that for q even the $(n-1)$-dimensional space $\tau \cap \tau_i \cap \tau_j$ is contained in the $(3n-1)$-dimensional space $\zeta\zeta_i\zeta_j$.

Assume again that the TGQ $T(O)$ has property (G) at the flag $((\infty), \zeta)$. We consider now the translation dual $T(O^*)$ of $T(O)$. We have $O^* = \{\tau, \tau_1, \ldots, \tau_{q^{2n}}\}$, ζ is the tangent space of O^* at τ, and ζ_i is the tangent space of O^* at τ_i, $i = 1, 2, \ldots, q^{2n}$. Also, in the space $PG(4n-1, q)$ of O^* the $(3n-1)$-dimensional space $PG(3n-1, q) = \tau\tau_i\tau_j$ $(i \neq j)$, with τ_i and τ_j any two distinct elements of O^*, contains exactly $q^n + 1$ elements of O^*. In the $PG(4n, q)$ containing $T(O^*)$ we now consider a $PG(3n, q)$ containing $PG(3n-1, q)$ but not contained in $PG(4n-1, q)$. Then the point (∞) together with the points and lines of $T(O^*)$ in $PG(3n, q)$ define a TGQ $T(\tilde{O}^*)$ of order $s = q^n$, where $\tilde{O}^*$ is the set of all elements of O^* in $PG(3n-1, q)$ (see [11, Section 8.7]).

THEOREM 4.3.4. *The TGQ $T(O^*)$ contains at least $s^3 + s^2$ subquadrangles (TGQ) $T(\tilde{O}^*)$ of order s. The projective plane π_τ of order s defined by the regular line τ of $T(\tilde{O}^*)$ is Desarguesian. It follows that for s odd, each of the $s^3 + s^2$ GQ $T(\tilde{O}^*)$ is isomorphic to the classical GQ $Q(4, s)$.*

Proof. Project the elements of $O^* - \{\tau\}$ from τ onto a $(3n-1)$-dimensional space η skew to τ. The projection of τ_i from τ onto η is denoted by τ'_i, $i = 1, 2, ..., q^{2n}$. Then $\tau'_1 \cup \tau'_2 \cup \cdots \cup \tau'_{q^{2n}}$ is $\eta - \zeta'$, where ζ' is the $(2n-1)$-dimensional space $\zeta \cap \eta$. Further, the set of all intersections $\tau\tau_i\tau_j \cap \eta$, $i \neq j$, is denoted by $\mathscr{B}$. Then with respect to inclusion the elements of $\mathscr{P} = \{\tau'_1, ..., \tau'_{q^{2n}}\}$ and $\mathscr{B}$, are the points and lines, respectively, of a $2-(q^{2n}, q^n, 1)$ design $\mathscr{A}$, that is, of an affine plane of order $q^n = s$. If ϱ' and ϱ'' are distinct parallel lines of $\mathscr{A}$, then $\varrho' \cap \varrho''$ is necessarily an $(n-1)$-dimensional subspace of ζ'. Hence all lines of $\mathscr{A}$ parallel to ϱ' contain a common $(n-1)$-dimensional subspace of ζ'. These $q^n + 1$ (distinct) subspaces of ζ' are denoted by $v_0, v_1, ..., v_{q^n}$. If for some $i \neq j$ $v_i \cap v_j \neq \emptyset$, say $x \in v_i \cap v_j$, then lines ϱ' and ϱ'' of $\mathscr{A}$ containing respectively v_i and v_j, do not intersect in a point of $\mathscr{A}$, so are parallel, that is $v_i = v_j$, a contradiction. Hence $\{v_0, v_1, ..., v_{q^n}\}$ is a partition (a spread) of ζ'. It is clear that the elements $v_0, v_1, ..., v_{q^n}$ and ζ' extend $\mathscr{A}$ to a projective plane Π of order q^n. Now by a theorem of Segre [14] such a plane Π is always Desarguesian; that is, in η there are n planes $\pi_1, \pi_2, ..., \pi_n$ over $GF(q^n)$, which are conjugate with respect to the nth extension $GF(q^n)$ of $GF(q)$ and which generate η, such that each plane π_i has a point in common with each element of the pointset of Π (any point of Π is generated by a point of π_1 and its conjugates).

Now we consider the projective plane π_τ of order $s = q^n$ defined by the regular line τ of $T(\tilde{O}^*)$. Let $\tilde{O}^*$ be defined by $PG(3n-1, q) = \tau\tau_i\tau_j$ and let $T(\tilde{O}^*)$ be contained in $PG(3n, q)$. Further, let $\tau\tau_i\tau_j \cap \eta = \bar{\eta}$ and let $\bar{\eta} \subset PG(2n, q) \subset PG(3n, q)$, with $PG(2n, q) \not\subset \tau\tau_i\tau_j$. The points of Π in $\bar{\eta}$ form the spread S. Identifying each line of type (a) of $T(\tilde{O}^*)$ concurrent with τ with its intersection with $PG(2n, q)$, each line of type (b) of $T(\tilde{O}^*)$ distinct from τ with its projection from τ onto $\bar{\eta}$, and τ with $\zeta \cap \bar{\eta}$, we see that π_τ is isomorphic to the projective translation plane $\pi(S)$ defined by the spread S (cf. [3]). By the end of the previous section, S is a regular $(n-1)$-spread (see [5, Sect. 25.6]) of $\bar{\eta}$. Hence $\pi(S)$, and so π_τ, is Desarguesian (see [5, Sect. 25.7]).

Finally, suppose that s is odd. As the spread S is regular, it follows from a result by Casse, Thas, and Wild (see [11, 8.7.7]) that $T(\tilde{O}^*)$ is isomorphic to the classical GQ $Q(4, s)$ arising from the nonsingular quadric of $PG(4, s)$. ∎

LEMMA 4.3.5. *Suppose that $T(\tilde{O}^*)$ is isomorphic to the classical GQ $Q(4, s)$, with $\tilde{O}^*$ being contained in the space $PG(3n-1, q)$. Then $\tilde{O}^*$ can be extended to an $(n-1)$-spread $\tilde{S}$ of $PG(3n-1, q)$ having the property that any $(2n-1)$-dimensional space generated by two distinct elements of $\tilde{S}$ contains exactly $q^n + 1$ elements of $\tilde{S}$. These $(2n-1)$-dimensional spaces* *together with the elements of $\tilde{S}$, and with containment as incidence, are the lines and points of a Desarguesian plane $PG(2, q^n)$.*

Proof. The last part of the statement follows from a theorem of Segre [14]. The first part easily follows from:

(a) Let $x \sim (\infty) \sim y$, $x \not\sim y$. If $u, v \in \{x, y\}^\perp$ with $u \neq (\infty) \neq v$, $(\infty) \mathrm{I}\, L$ with $x \not\mathrm{I}\, L$, $y \not\mathrm{I}\, L$, and $u \mathrm{I}\, U \sim L$, $v \mathrm{I}\, V \sim L$, then each point of $\{x, y\}^\perp$ is incident with a line of $\{U, V\}^\perp$ (or, equivalently $\{U, V\}^{\perp\perp}$).

(b) Consider $\{L_1, N_1\}^\perp$, $\{L_2, N_2\}^\perp$, with $(\infty) \mathrm{I}\, L_1$, $(\infty) \mathrm{I}\, L_2$, $(\infty) \not\mathrm{I}\, N_1 \not\sim L_1$, $(\infty) \not\mathrm{I}\, N_2 \not\sim L_2$, $L_2 \notin \{L_1, N_1\}^{\perp\perp} \cup \{L_1, N_1\}^\perp$. Then the lines of $\{L_1, N_1\}^\perp$ and $\{L_2, N_2\}^\perp$ are incident with $q^n + 1$ common points. The set of these points is denoted by D. If u, v are two distinct points of D, with $u \neq (\infty) \neq v$, if $(\infty) \mathrm{I}\, L$ with $L \notin \{L_i, N_i\}^{\perp\perp} \cup \{L_i, N_i\}^\perp$, $i = 1, 2$, and if $u \mathrm{I}\, U \sim L$, $v \mathrm{I}\, V \sim L$, then each point of D is incident with a line of $\{U, V\}^\perp$ (or, equivalently $\{U, V\}^{\perp\perp}$).

(c) From (a) and (b) follows that $\tilde{O}^*$ can be extended to an $(n-1)$-spread $\tilde{S}$ of $PG(3n-1, q)$, that each conic of Q through (∞) is in $T_2(\tilde{O}^*)$ of the form $\{(\infty)\} \cup (PG(n, q) - PG(3n-1, q))$ with $PG(n, q)$ an n-dimensional subspace of $PG(3n, q)$ intersecting $PG(3n-1, q)$ in an element of $\tilde{S} - \tilde{O}^*$, that each $(2n-1)$-dimensional subspace of $PG(3n-1, q)$ containing at least two elements of $\tilde{O}^*$ contains exactly $q^n + 1$ elements of $\tilde{S}$, and that each $(2n-1)$-dimensional subspace of $PG(3n-1, q)$ containing exactly one element of $\tilde{O}^*$ and being disjoint from all other q^n elements of $\tilde{O}^*$ contains exactly $q^n + 1$ elements of $\tilde{S}$.

(d) From (c) follows that each elliptic quadric on Q through (∞) is in $T_2(\tilde{O}^*)$ of the form $\{(\infty)\} \cup (PG(2n, q) - PG(3n-1, q))$ with $PG(2n, q)$ a $2n$-dimensional subspace of $PG(3n, q)$ intersecting $PG(3n-1, q)$ in a $(2n-1)$-dimensional space containing exactly $q^n + 1$ elements of $\tilde{S} - \tilde{O}^*$. The number of these $(2n-1)$-dimensional spaces is equal to $(q^n - 1)\, q^n/2$. ▮

COROLLARY 4.3.6. *The dual $\widehat{\mathcal{S}(F)}$ of a TGQ $\mathcal{S}(F)$ of Kantor* (*cf.* 4.2) *contains at least $s^3 + s^2$ classical subquadrangles $Q(4, s)$. Equivalently, $\mathcal{S}(F)$ contains at least $s^3 + s^2$ classical subquadrangles $W(s)$. Any Roman TGQ of Payne* (cf. 4.2) *contains at least $s^3 + s^2$ classical subquadrangles $Q(4, s)$.*

Let $\mathcal{S}$ be a GQ of order (s, s^2), s even, having property (G) at the flag (x, L) and suppose that $\mathcal{S}$ is a TGQ with base point x. Further, let x be a center of the triad $\{y, z, u\}$, with $y \mathrm{I}\, L$, and let $\mathcal{S}'$ be the subquadrangle of order s containing $\{y, z, u\}^\perp$ and $\{y, z, u\}^{\perp\perp}$. As L is a regular line of the TGQ $\mathcal{S}$, it is also a regular line of $\mathcal{S}'$.

THEOREM 4.3.7. *For s even, the projective plane π_L arising from the regular line L of $\mathcal{S}'$ is Desarguesian.*

Proof. By Theorem 4.3.2 the translation dual $T(O^*)$ of $T(O) \cong \mathcal{S}$ has property (G) at the flag $((\infty), \tau)$ with τ the tangent space of O at ζ. So by Theorem 4.3.1 the $(3n-1)$-dimensional space $PG(3n-1, q) = \zeta\zeta_i\zeta_j$, with ζ_i and ζ_j any two distinct elements of $O - \{\zeta\}$, contains exactly $q^n + 1$ elements of O. If $\tilde{O}$ is the set of these $q^n + 1$ elements, then $\mathcal{S}'$ is of type $T(\tilde{O})$, where $T(\tilde{O})$ is defined by $\tilde{O}$ and a $3n$-dimensional subspace $PG(3n, q)$ containing $PG(3n-1, q)$ but not contained in $PG(4n-1, q)$ (cf. proof of 4.3.2). By Theorem 4.3.4 the projective plane π_ζ of order s defined by the regular line ζ of $T(\tilde{O})$ is Desarguesian; hence π_L is Desarguesian. ∎

5. Ovoids of the Generalized Quadrangle $Q(4, s)$, s Odd

5.1. *Definition and Known Results*

An ovoid $\mathcal{O}$ of the GQ $Q(4, s)$, s odd, is a set of $s^2 + 1$ points, no two of which are collinear; equivalently $\mathcal{O}$ has a point in common with each line of $Q(4, s)$.

If $Q \subset PG(4, s)$ and if the hyperplane $PG(3, s)$ intersects Q in an elliptic quadric $\mathcal{O}$, then $\mathcal{O}$ is an ovoid of $Q(4, s)$. For s an even power of 2 no other ovoids of $Q(4, s)$ are known. For $s = 2^{2h+1}$, $h \geqslant 1$, one other type of ovoid is known; its projection from the nucleus n of Q onto a hyperplane not containing n, is the Tits ovoid; see [15]. For s odd, Kantor [7] constructed two types of nonclassical ovoids: (i) type $\mathcal{K}_1$, here $s = p^h$, p an odd prime and $h > 1$, and these ovoids are characterized by the property that they are in just one way the union of s conics which are mutually tangent at a common point (see also [16]), and (ii) type $\mathcal{K}_2$, here $s = 3^{2h+1}$, $h \geqslant 1$, and each such ovoid arises from a Ree–Tits ovoid on a nonsingular quadric in projective 6-space.

With each ovoid of $Q(4, s)$ there corresponds a spread, which is a partition by lines, of the dual GQ $W(s)$. And with a spread of $W(s)$ there corresponds, by the standard argument, a translation plane $\mathcal{P}$ of order s^2. If $\mathcal{O}$ is the elliptic quadric, then $\mathcal{P}$ is Desarguesian. If $\mathcal{O}$ is of Tits type, then $\mathcal{P}$ is the Lüneburg plane; see [15]. If $\mathcal{O}$ is of Kantor type $\mathcal{K}_1$, then $\mathcal{P}$ is a Knuth semifield plane; see [7].

5.2. *Subquadrangles and Ovoids*

Let $\mathcal{S}'$ be a subquadrangle of order (s, t') of the GQ $\mathcal{S}$ of order (s, t), $t' < t$. If z is a point of $\mathcal{S}$ which is not contained in $\mathcal{S}'$, then the $1 + st'$ points of $\mathcal{S}'$ collinear with z form an ovoid $\mathcal{O}$ of $\mathcal{S}'$; see [11, 2.2.1]. For example, if $\mathcal{S} = Q(5, s)$ and $\mathcal{S}' = Q'(4, s)$, with $Q' \subset Q$, then $\mathcal{O}$ is an elliptic quadric on Q'.

5.3. *Ovoids of* $T(\tilde{O}^*) \cong Q(4, s)$ *Arising from* $T(O^*)$

Assume that the TGQ $T(O)$, with $O = \{\zeta, \zeta_1, ..., \zeta_{q^{2n}}\}$ consisting of $q^{2n}+1$ $(n-1)$-dimensional subspaces in $PG(4n-1, q)$, has property (G) at the flag $((\infty), \zeta)$. Let $T(O^*)$ be the translation dual of $T(O)$. Then $O^* = \{\tau, \tau_1, ..., \tau_{q^{2n}}\}$, with τ the tangent space of O at ζ and τ_i the tangent space of O at ζ_i, $i = 1, 2, ..., q^{2n}$. Further, let q be odd. Then by Theorem 4.3.4 the GQ $T(O^*)$ has s^3+s^2, $s = q^n$, subquadrangles $T(\tilde{O}^*)$ isomorphic to $Q(4, s)$.

If z is a point of $T(O^*)$ which is not contained in the subquadrangle $T(\tilde{O}^*)$, then the corresponding ovoid of $T(\tilde{O}^*)$ will be denoted by $\mathcal{O}(z)$.

THEOREM 5.3.1. *If $T(O)$ is a TGQ of Kantor type (including the classical GQ $Q(5, s)$), then $\mathcal{O}(z)$ is always an elliptic quadric or of Kantor type $\mathcal{K}_1$.*

Proof. Let $T(O)$ be a TGQ of Kantor type (cf. 4.2). Then $T(O) \cong T(O^*)$. Let $\tilde{O}^*$ be contained in $PG(3n-1, q)$ and assume that $\tilde{\mathcal{O}}^* = \{\tau, \tau_1, ..., \tau_{q^n}\}$. Then τ_i (resp. τ) is contained in exactly one $(2n-1)$-dimensional space $\tilde{\zeta}_i$ (resp. $\tilde{\zeta}$) of $PG(3n-1, q)$, having an empty intersection with each element of $\tilde{O}^* - \{\tilde{\zeta}_i\}$ (resp. $\tilde{O}^* - \{\tilde{\zeta}\}$), $i = 1, 2, ..., q^n$. Since $T(O^*)$ has property (G) at the flag $((\infty), \tau)$, the $(n-1)$-spaces $\tilde{\zeta} \cap \tilde{\zeta}_i$ are exactly the $(n-1)$-spaces $\tilde{\zeta} \cap \zeta_j$, $i = 1, 2, ..., q^n$, and $j = 1, 2, ..., q^{2n}$.

First, let z be a point of type (ii) of $T(O^*)$ which is not contained in $T(\tilde{O}^*)$. Then $\mathcal{O}(z) = \{(\infty)\} \cup (PG(2n, q) - PG(3n-1, q))$, where $PG(2n, q)$ is a subspace of the space $PG(3n, q)$ in which $T(\tilde{O}^*)$ is defined and with $PG(2n, q) \cap PG(3n-1, q)$ of the form $\zeta_j \cap PG(3n-1, q)$, $j \in \{q^{n+1}, ..., q^{2n}\}$. Let us now identify $T(\tilde{O}^*)$ with the GQ $T_2(C)$, where C is a conic of $PG(2, s)$. Then in $T_2(C)$ the ovoid $\mathcal{O}(z)$ consists of (∞) and the points of type (i) on s lines of $PG(3, s)$ containing a common point on a tangent line of C. So in $Q(4, s) \cong T_2(C)$, $\mathcal{O}(z)$ consists of s conics which are mutually tangent at a common point (∞).

Next, let z be a point of type (i) of $T(O^*)$ which is not contained in $T(\tilde{O}^*)$. Then $\mathcal{O}(z)$ consists of $s+1$ points of type (ii) and the s^2-s points u_j, with $\{u_j\} = \tau_j z \cap PG(3n, q)$ and $j = s+1, ..., s^2$. The s^2-s spaces τ_j, $j = s+1, ..., s^2$, are distributed over $s-1$ $(3n-1)$-dimensional spaces containing $\tilde{\zeta}$. Hence the s^2-s points u_j are distributed over $s-1$ $2n$-dimensional spaces $\pi_1, ..., \pi_{s-1}$ containing $\tilde{\zeta}$. If u_j and $u_{j'}$, $j \neq j'$, belong to a common space π_k, then $u_j u_{j'}$ is skew to τ and $\tau_j \tau_{j'} \cap \tilde{\zeta}$ is one of the spaces $\tilde{\zeta} \cap \tilde{\zeta}_i$, $i = 1, 2, ..., q^n$ (cf. Lemma 4.3.5). It follows that for any three distinct points u_j, $u_{j'}$, and $u_{j''}$ of a common space π_k, the points $u_j u_{j'} \cap \tilde{\zeta}$, $u_j u_{j''} \cap \tilde{\zeta}$, $u_{j'} u_{j''} \cap \tilde{\zeta}$ lie in different spaces $\tilde{\zeta} \cap \tilde{\zeta}_i$. In $T_2(C)$ the ovoid $\mathcal{O}(z)$ consists of $s+1$ points of type (ii) and s^2-s points u_j of type (i) distributed over $s-1$ planes $\pi_1, ..., \pi_{s-1}$ containing a common tangent line Z of C. The common point of Z and C is denoted by t. By a preceding argument the point t together with the s points u_j in a common plane π_k constitute a set of $s+1$

points no three of which are collinear, that is, an oval of π_k. As s is odd this oval is a conic C_k (see 8.2.4 of [4]). Clearly Z is the tangent line of C_k at t. In $T_2(C)$ all points of $C_k - \{t\}$ are collinear with the point π_k of type (ii). Now it is clear that in $Q(4, s) \cong T_2(C)$ $\mathcal{O}(z)$ consists of $s+1$ points collinear with (∞) and of the $s^2 - s$ points of $s-1$ sets $\bar{C}_k$, where $C'_k = \bar{C}_k \cup \{z_k\}$ is a conic and the points $z_1, z_2, ..., z_{s-1}$ are on a line T through (∞). All points of C'_k are collinear with a point w_k on T, and hence also with a second point w'_k. If $z_1 \notin \mathcal{O}(z)$, then, as each line through z_1 contains exactly one point of $\mathcal{O}(z)$, the point z_1 is necessarily one of the points w_k, say w_2. As $v \in \bar{C}_2$ is collinear with no point of $\bar{C}_1$, the line $z_1 v$ contains the point w'_1. Hence w'_1 is on each line $z_1 v$, a contradiction. Consequently $z_1 = z_2 = \cdots = z_{s-1} = \bar{z}$ is the point of $\mathcal{O}(z)$ on the line T. As w'_1 is collinear with no point of $\bar{C}_k$, $k \neq 1$, we necessarily have $w'_1 \sim w'_k$. It follows that $w'_1, w'_2, ..., w'_{s-1}$ are on a common line $\bar{W}$ through $\bar{z}$. Hence the planes of the conics C'_1, C'_2 are in the polar space η of $w_1 w'_1 \cap w_2 w'_2$ with respect to Q. So C'_1, C'_2 are on the elliptic quadric $\eta \cap Q$, and since $C'_1 \cap C'_2 = \{\bar{z}\}$ they are tangent at $\bar{z}$. Consequently $C'_1, ..., C'_{s-1}$ are mutually tangent at $\bar{z}$. Clearly no two of the points $\bar{z}, w'_1, ..., w'_{s-1}$ coincide. Let w'_s be the remaining point of the line $\bar{W}$. No point of $\bar{C}_1 \cup \bar{C}_2 \cup \cdots \cup \bar{C}_{s-1}$ is collinear with w'_s, and so w'_s is collinear with the $s+1$ points of $\mathcal{O}(z)$ in $(\infty)^\perp$. It follows that the set C'_s of all points of $\mathcal{O}(z)$ collinear with (∞) is the conic $\{(\infty), w'_s\}^\perp$. Now an argument used before shows that C'_s and C'_k, $k = 1, 2, ..., s-1$, are mutually tangent at $\bar{z}$. Consequently the conics $C'_1, C'_2, ..., C'_s$ are mutually tangent at $\bar{z}$; that is, $\mathcal{O}(z)$ is an elliptic quadric or of Kantor type $\mathcal{K}_1$. ∎

Conjecture 5.3.2. If $T(O^*)$ is a Roman GQ and $q = 3^r$ with $r > 2$, then for each point z of type (i) not contained in $T(\tilde{O}^*)$, the ovoid $\mathcal{O}(z)$ is neither an elliptic quadric nor of Kantor type $\mathcal{K}_1$. Then it is clear that for r even, the ovoid $\mathcal{O}(z)$ is new.

Note added in proof. This conjecture is proved in: J. A. THAS AND S. E. PAYNE, Spreads and ovoids in finite generalized quadrangles, *Geom. Dedicata*, to appear. For any $r \geqslant 3$ the avoid, and also the corresponding plane, are new.

Part II. The goal of Part II is the classification of all translation generalized quadrangles of order (s, s^2), $s > 1$, satisfying property (G) at some flag $((\infty), L)$. At this moment many partial results are already obtained.

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